Arising of magnetic walls in the vicinities of the Frèedericksz transition

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In this paper, the dynamical formation of twist-bend periodic walls at the neighborhoods of the Frèedericksz critical point of a nematic sample of liquid crystals is studied. The theory that describes the arising of these periodic structures affirms that the mode with the fastest initial growth will fix the observed properties of these patterns. But, just above the Frèedericksz threshold there is a region where this leading mode becomes null and, therefore, a homogeneous bending of the director may be detected. This prediction was not confirmed by the experiment, and walls with very well defined wavelength were found. We will show here that the fastest growing mode cannot be defined around the Frèedericksz threshold and, therefore, a new way to compute the observed periodicity must be formulated. We assume that the observed wall results from a sum of a continuum and nonsharp distribution of modes in which the null mode is at the center.

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I. INTRODUCTION

It is well known that, in nematic liquid crystals (NLCs), the coupling between the bending of the director and the movement of the nematic fluid can produce highly symmetric patterns. The so-called magnetic walls are examples of these well-studied textures. They make the transition between adjacent symmetrical distorted regions of nematic phase and are usually found when, under appropriated conditions, the NLC is submitted to an external magnetic field [1] greater than the Frèedericksz threshold [2]. Once formed, these patterns are not stable. It has been observed that, for high magnetic fields, they decline, losing each of their attributes: extremely regular, well-defined, and seemingly one-dimensional periodic structures [3–6].

The theory that first described the formation of these structures in NLCs stated that, for high magnetic fields, the coupling between the bending of the director and the concomitant flow of the nematic fluid could exponentially amplify some random thermal fluctuations [7–9]. In this scenario, as soon as the fluid flow stops, the fate of these structures begins and the dynamically coupled process acting during the fluid flow must be responsible for the outstanding seemingly one-dimensional and periodic character exhibited by these walls [10–12]. With the use of the anisotropic dynamical properties of NLCs, Lonberg *et al.* [8] have supposed that the observed periodicity results from a selection mechanism that amplifies to macroscopic scale some well-defined thermal fluctuations. According to this model, all random thermal modes are amplified. But, its central concept affirms that the final observed profile of these patterns is determined, in the beginning of the process, by the modes having the fastest initial amplification. From now on, this mechanism will be referred to as the leading mode principle.

Nevertheless, there is some theoretical and experimental evidence indicating that, whenever a good image of the beginning of these structures is provided, this principle may not give a full explanation of the pattern observed on the magnetic walls [5,9,13]. In a recent paper, some of us have found that, in the neighborhoods of the Frèedericksz critical point, the comparison between the predictions of the leading mode principle and experimental facts is in irreconcilable disagreement [14,6]. That is, the leading mode principle predicts that when the Frèedericksz threshold is approached, the coherent internal motion of the nematic material becomes smaller and smaller and there is a point—H_{lm}—greater than the Frèedericksz critical point—H_F—below which there is no more induced motion of the nematic material. The region $H_F<H<H_{lm}$ is known as the forbidden region. So, according to the usual interpretation for the arising of magnetic walls in nematic liquid crystals, in the forbidden region, the torque of the external field on the nematic molecules produces a uniform alignment, and the periodic walls are not detected. Nevertheless, in an experimental investigation, using the twist-bend geometry [1,8,15], we found that when the Frèedericksz threshold is approached, the emergence of these structures continues [6] and, no matter how close the Frèedericksz critical point is approached, a homogeneous alignment of the director has never been found. Moreover, it was observed that the time spent in the construction of these periodic walls diverges as the critical point is approached. This has not been considered by Lonberg in his calculations concerning the neighborhoods of the critical point since, according to them [6], the formation of any magnetic wall would encompass a finite time interval, and only with the formation of a homogeneous bending—produced at the neighborhoods of the Frèedericksz critical point—would an infinite time interval be expected. Therefore, the predictions of the theory based on the leading mode principle were not confirmed and a forbidden region was not detected by the experiment.

The aim of this paper is to study the pattern formation in the region above the Frèedericksz threshold where the current model, based on the leading mode principle, seems to be inadequate. A careful analytical study of the nonlinearities involved in the pattern formation at the neighborhoods of the Frèedericksz threshold will be undertaken. It will be shown that, in the neighborhoods of the critical point, a unique lead-

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ing mode cannot be clearly defined. In fact, there is a set of neighboring modes, continuously distributed, that grow practically at the same rate. The result of this collective growth is not a homogeneous bending of the director, as predicted by the leading mode principle, but is rather a set of very well defined periodic walls whose formation tends to encompass an infinite time interval when the Frèedericksz threshold is approached.

This work is divided into two main sections. In the next section, we will formulate a nonlinear mathematical model for the arising of the periodic walls. With this formulation, we will be able to obtain the final amplitude of each Fourier component of the magnetic wall. In the final section, we will use the results of Sec. II to compute the periodicity and the time spent with the formation of the periodic walls in the forbidden region.

II. FUNDAMENTALS

In order to obtain workable equations for a description of the arising of the magnetic walls, some approximations will be done on the set of equations describing the hydrodynamics of the nematic material, the so-called Eriksen-Leslie-Parodi (ELP) approach [16–18]. Normally [8–12,19,20], these approximations are conducted in such a way that the final equations become linear. The predicted immediate effect of this procedure is an unbounded exponential growing of the walls' amplitude and, as a consequence, the saturated behavior of the system cannot be studied. As, by definition, the leading mode should be determined at the initial moments of the process, this practice is sufficient to select the initial fastest growing mode. However, the patterns to be described in this work do not arise immediately after the initiation of the magnetic field. Their emergence lasts a long time and, therefore, the linear approach is not acceptable. So, along the approximations to be described below, we will avoid the complete linearization of the equations. The choice of the nonlinear terms to be retained will be guided by the criterion that stresses that only those ELP equation terms that are decisive for a saturated profile of the walls' amplitude are important in the long behavior of these structures. Furthermore, even being described by nonlinear equations, we will show that, in the forbidden region, the coupling between different modes is negligibly small and need not be considered.

To study the physics of the periodic walls at the neighborhoods of the Frèedericksz transition, a NLC sample inside a microslide glass with dimensions (a,b,d) that satisfy the relation $a \gg b \gg d$ will be considered. The director is initially uniformly aligned along the $\vec{e}_x$ direction and an external controlled magnetic field H is applied along the $\vec{e}_y$ direction. When this is done, there is a competition between the magnetic susceptibility, which tends to align the director along the direction of the magnetic field, and the elastic energy, which tends to produce a director orientation consistent with its orientation at the surface of the sample. When H is greater than the Frèedericksz threshold H_F, the magnetic susceptibility overcomes the elastic resistance of the medium, and the director begins to bend towards the direction of the external magnetic field, trying to become parallel or antiparallel to it. Due to this symmetry breaking, the profile of the director becomes inhomogeneous and some well-defined and symmetric patterns fill the sample [1,2,21,22]. These patterns have the aspect of a set of one-dimensional structures—magnetic walls—periodically distributed along the $\vec{e}_x$ direction.

The first approximation that will be done for the study of these structures assumes that the director field maintains its initial planar geometry during its rotation provoked by the external magnetic field. That is, the sample is prepared in such a way that, initially, the orientation of the external magnetic field is perpendicular to the director direction at all points in the sample. Hence, these two directions—the magnetic field and the director direction—define a plane on the sample and, as there is not an external torque pushing the director out of this plane, it can be assumed that, during the director rotation, such a plane maintains its orientation. Furthermore, as stated above, the width d of the sample along the $\vec{e}_z$ direction is so small when compared with the other dimensions of the sample that it can be assumed that the elastic interaction (which is proportional to $1/d^2$) will immobilize any director motion along this direction. These planar conditions can be used to obtain two important simplifications on the hydrodynamics equations. The first one is the imposition that the director does not acquire any component along the $\vec{e}_z$ direction [8],

$$n_x = \cos\theta(x,y,z), \quad n_y = \sin\theta(x,y,z), \quad n_z = 0. \tag{1}$$

The second one is the assumption that the fluid velocity $\vec{V}$ does not have components along the $\vec{e}_z$ direction, that is,

$$V_z = 0. \tag{2}$$

These two conditions are equivalent to assuming that the dynamical equations for this problem are bidimensional. That is, when considering the equations of the ELP approach, only its components along the plane defined by the axis $\vec{e}_x$ and $\vec{e}_y$ must be considered.

In the ELP approach [1,16–18], the time evolution of the director direction and the motion of the nematic material are given by a set of differential equations composed by the anisotropic version of the Navier-Stokes equation, the balance of torques equation, and the equation of continuity. We begin our study with the continuity equation that, as the system is assumed to be incompressible, takes the form [23,24]

$$\partial_\alpha V_\alpha = 0. \tag{3}$$

One important characteristic of the geometry of the magnetic walls can be understood from this equation. It is an experimental fact that the component V_x of the velocity is relevant only at the borders of the sample [8,11,12]. Consequently, as the magnetic walls are observed far away from the borders, one can make $V_x = 0$ on the continuity equation. As the system is dynamically bidimensional, the continuity equation gives

$$\partial_y V_y = 0. \tag{4}$$

for the region where the walls are observed. Essentially, this equation demonstrates that as long as the problem can be considered bidimensional and there is no fluid motion along the $\vec{e}_x$ direction, the velocity of the nematic fluid will remain constant along the $\vec{e}_y$ direction. The characteristic form of the stripes observed in the magnetic walls is an immediate consequence of this equation.

The Navier-Stokes equation is given by

$$\rho\left(\frac{\partial V_\alpha}{\partial t} + V_\beta \frac{\partial V_\alpha}{\partial x_\beta}\right) = \frac{\partial}{\partial x_\beta}(-p\,\delta_{\alpha\beta} + \sigma_{\beta\alpha}), \tag{5}$$

where ρ is the density of the system, V_α is the α component of the velocity, p is the pressure, and $\sigma_{\beta\alpha}$ is the associated anisotropic stress tensor, which depends on the velocity $\vec{V}$ of the fluid, the bending of the director θ, and its time variation rate $\dot{\theta}$ [1,23,24].

The approximation assumed above allows us to consider the problem as bidimensional. So, the components of the Navier-Stokes describing the motion of the nematic fluid along the $\vec{e}_x$ and $\vec{e}_y$ directions are sufficient to describe the walls' phenomenology. Hence, the pressure p can be eliminated from these equations by subtracting one of the components of these equations from another [11]. Furthermore, we will consider that the velocity of the matter in the sample is such that we can neglect the nonlinear term $V_\beta \partial_\beta V_\alpha$. Thus,

$$\rho \frac{d}{dt}(\partial_x V_y - \partial_y V_x) = \partial_x^2 \sigma_{xy} - \partial_y^2 \sigma_{yx} + \partial_x \partial_y (\sigma_{yy} - \sigma_{xx}) + \partial_z(\partial_x \sigma_{zy} - \partial_y \sigma_{zx}). \tag{6}$$

To further simplify this equation, we must remember that the characteristic time involved in the phenomenon that we are studying is so long [6] that the viscosity becomes the dominant dynamical parameter of the nematic fluid flow and, therefore, any inertial term can be neglected. Furthermore, using the expressions for the viscosity tensor given, for example, in Refs. [23,24], and taking into account Eq. (4), we get

$$c_1(\theta)\partial_x^2(\partial_x V_y) + c_2(\theta)\partial_x V_y + c_3(\theta)\partial_t \theta + c_4(\theta)\partial_x^2(\partial_t \theta) = 0, \tag{7}$$

where

$$\begin{aligned} c_1(\theta) &= \eta_3 + (\eta_2 - \eta_3)\sin^2\theta + (\eta_1 - \eta_3 + 2\eta_3 \sin^2\theta)\cos^2\theta, \\ c_2(\theta) &= -\frac{1}{2}\left(\frac{\pi}{d}\right)^2 [\eta_2 + \eta_3 + (\eta_3 - \eta_2)\cos 2\theta], \\ c_3(\theta) &= \gamma_1 \partial_x^2 \theta \sin 2\theta, \\ c_4(\theta) &= -\frac{1}{2}\gamma_1(1 + \cos 2\theta), \end{aligned} \tag{8}$$

and η_1, η_2, and η_3 are the Miesowicz's coefficients, γ_1 is the viscosity's coefficients related to the rotation of the director (it was assumed that $\gamma_2 \approx -\gamma_1$), and along the direction $\vec{e}_z$ it was assumed that $\partial_z^2 \theta = -(\pi/d)^2 \theta$.

Moreover, we also consider that the walls' periodicity along the $\vec{e}_x$ direction allows us to Fourier-decompose $\partial_x V_y$ and θ. That is,

$$\begin{aligned} \partial_x V_y &= \sum_n a_n \cos\left(n \frac{2\pi}{L} x\right), \\ \theta &= \sum_n b_n \cos\left(n \frac{2\pi}{L} x\right) \Rightarrow \partial_t \theta = \sum_n \dot{b}_n \cos\left(n \frac{2\pi}{L} x\right), \end{aligned} \tag{9}$$

where $n = 0,1 \cdots L/2$ and $\dot{b}_n = \partial_t b_n$. So, from these equations it follows that

$$a_n = -\frac{c_3(\theta) - k^2 c_4(\theta)}{c_2(\theta) - k^2 c_1(\theta)} \dot{b}_n. \tag{10}$$

So, each Fourier component of $\partial_t \theta(\dot{b}_n)$ induces a shearing motion of the nematic material, inducing a non-null value to the Fourier components of $\partial_x V_y(a_n)$. It is important to observe that, through the functions c_1, c_2, c_3, and c_4 given in Eq. (8), Eq. (10) is strongly dependent on θ and, therefore, it is also strongly dependent on b_n, causing the coupling between the different modes. At this point, the usual approach [8] restricted the analysis of the walls' formation to the θ-independent term of this equation. This procedure predicts an unbounded growing for θ and, consequently, does not predict a saturated profile for the walls. But, when we consider the next-order term, we obtain

$$a_k = R_0\left(1 - \frac{\theta^2}{\varphi_o^2}\right)\dot{b}_k, \tag{11}$$

where

$$R_0 = \frac{\gamma_1 \tilde{k}^2}{\eta_3 + \tilde{k}^2 \eta_1} \quad \text{and}$$

$$\varphi_o^2 = \frac{\eta_3 + \tilde{k}^2 \eta_1}{(\eta_2 + 2\eta_3) + \tilde{k}^2(2\eta_1 + \eta_2 + 2\eta_3)}, \tag{12}$$

and $\tilde{k}^2 = (kd/\pi)^2$ is the reduced wave vector. This equation shows that for each $\tilde{k}$, there is an angle, φ_o, above which the Fourier component of the bending of the director, $\dot{b}_k$, no longer induces a non-null value to the corresponding Fourier component, a_k, of the shearing, $\partial_x V_y$.

As the above equation is not sufficient to find independent solutions for V_y and θ, another equation is needed. This equation is the balance of torques equation [1,23,24] that, for the planar geometry defined by Eqs. (1) and (2), assumes the following form:

$$\gamma_1 \partial_t \theta = \gamma_1 n_x^2 (\partial_x V_y) + K_{33}[\partial_x^2 \theta + \partial_y^2 \theta] + K_{22} \partial_z^2 \theta + \chi_a H^2 n_x n_y . \qquad (13)$$

By substituting in this equation the value of the shear $\partial_x V_y$ obtained in Eq. (11) and making the changes

$$\tau = \frac{\chi_a H_c^2}{\gamma_1} t, \quad h^2 = \frac{H^2}{H_F^2}, \quad \tilde{K} = \frac{K_{33}}{K_{22}},$$

$$\chi_a H_F^2 = K_{22}\left(\frac{\pi}{d}\right)^2 + K_{33}\left(\frac{\pi}{b}\right)^2 \simeq K_{22}\left(\frac{\pi}{d}\right)^2, \qquad (14)$$

it is obtained that

$$\dot{b}_n = \frac{1}{\tau_o} b_n \left\{ 1 - \frac{b_n^2}{\theta_{\max}^2} \right\} - \frac{I_n}{\tau_o \theta_{\max}^2}, \qquad (15)$$

where

$$\tau_o(k) = \frac{(1-R_0)}{(h^2 - 1 - \tilde{K}\tilde{k}^2)} = \frac{\left(1 - \dfrac{\gamma_1 \tilde{k}^2}{\eta_3 + \tilde{k}^2 \eta_1}\right)}{(h^2 - 1 - \tilde{K}\tilde{k}^2)}, \qquad (16)$$

$$\theta_{\max}^2(\tilde{k}) = \frac{(h^2 - 1 - \tilde{K}\tilde{k}^2)}{\left[(h^2 - 1 - \tilde{K}\tilde{k}^2)\dfrac{R_0}{1-R_0}\left(1 + \dfrac{1}{\varphi_o^2}\right) + \dfrac{2}{3}h^2\right]}, \qquad (17)$$

and

$$I_n = \int_0^L \theta^3 \cos\left(n \frac{2\pi}{L} x\right) dx - b_n^3, \quad n = 0,1 \cdots \frac{L}{2}$$

describes the interaction between the different Fourier modes. Now, observe that the coupling of the term I_n is given by the product of the variables τ_o and $\theta_{\max}^2$, which determines the intensity of the interaction between the various modes. First, observe (this will be demonstrated in detail in the next section) that in the forbidden region the dependence of these variables on $\tilde{k}^2$ is so flat that it does select any mode and, therefore, in this region this coupling term can be considered constant. Furthermore, it is an experimental fact—which will be confirmed by our theory in the following—that, in this region, the time τ_o spent in the arising of the wall is exceedingly large, becoming infinite at the Frèedericksz threshold. So, this interacting term is a small constant that, in the forbidden region, can be neglected. Hence, our approximation for this problem considers that

$$\dot{b}_k = \frac{1}{\tau_o} b_k \left\{ 1 - \frac{b_k^2}{\theta_{\max}^2} \right\} \qquad (18)$$

gives the time evolution of each mode in the neighborhoods of the Frèedericksz threshold.

As can be seen, for example, in [13], Eq. (18) has the general form of the standard equations that describe the patterns formation in many physical systems. In this paper, it will be assumed that it describes the formation of the magnetic walls of NLCs in the neighborhoods of the Frèedericksz threshold. Their first-order term reproduces the already known results about the initial moments of the time evolution of b_k. Up to this order, an unbounded exponential growth for the amplitude b_k is predicted and, according to Lonberg *et al.* [8], the leading mode is the one that provides the fastest growth to this amplitude. Now, with the third-order term of Eq. (18), the unbounded growth of the walls' amplitude is prevented and, therefore, a study of the long behavior of these structures is possible.

Finally, Eq. (18) is integrable and its solution is given by

$$b_k(\tau) = \pm \theta_{\max}(\tilde{k}) \sqrt{\frac{A_o e^{[2\tau/\tau_o(\tilde{k})]}}{\theta_{\max}^2(\tilde{k}) + A_o e^{[2\tau/\tau_o(\tilde{k})]}}}, \qquad (19)$$

where A_o is an integration constant that may be fixed at $\tau = 0$.

This is the equation that gives the time evolution of each mode $\tilde{k}$. In the following section, the mathematical results of this section will be used to study the arising of magnetic walls at the neighborhoods of the Frèedericksz transition.

III. THE LEADING MODE

In this section, we will use Eq. (18) to examine the formation of magnetic walls at the neighborhoods of the Frèedericksz threshold and it will be shown that, in this region, the statement that says that there is a unique isolated mode determining the physical properties of the observed patterns is rather ambiguous and cannot be considered—even as an approximation—for the real physical situation. That is, a large set of states contributes equally to the formation of the magnetic walls in this region.

The leading mode principle states that the wave number, k_o, observed in the periodic walls can be determined by choosing the mode that gives the initial highest velocity to the amplitude of the periodic walls, b_k. So, according to the results of the preceding section, this mode must satisfy

$$\partial_k(\dot{b}_k) = 0 \quad \text{at} \quad \tau = 0. \qquad (20)$$

When this equation is applied to Eq. (18), it leads to

$$-\frac{\partial_k \tau_o}{\tau_o}\left\{1 - \frac{\theta_o^2}{\theta_{\max}^2}\right\} + 2\frac{\theta_o^2}{\theta_{\max}^3} \partial_k \theta_{\max} = 0, \qquad (21)$$

where θ_o stands for the initial distribution of the mode b_k that, as usual [1,9], can be found using the equipartition theorem. The behavior of the solution obtained with Eq. (21), when the magnetic field approaches the Frèedericksz threshold ($h^2 \simeq 1$), is now going to be discussed. An analytical computation shows that around $k^2 \approx 0$ the non-null and real solutions of Eq. (21) are given by

$$k^2 \approx -2\,\eta_3 \frac{3\,\eta_3[\,\gamma_1(1-h^2)+\widetilde{K}_{33}\eta_3]+\theta_o^2\gamma_1[3\,\eta_2(h^2-1)+\eta_3(11h^2-9)]}{P(\,\eta_1,\eta_2,\eta_3,\gamma_1,h^2)}$$

or

$$k^2 \equiv 0, \tag{22}$$

where $P(\,\eta_1,\eta_2,\eta_3,\gamma_1,h^2)$ is an awkward and non-null function that has no influence on the results presented below. From this equation, it is easy to see that in the whole interval

$$1 \leqslant h^2 \leqslant h_{lm}^2, \tag{23}$$

where

$$h_{lm}^2 = \left(\frac{H_{lm}}{H_F^2}\right)^2 = 1 + \frac{\eta_3(3\widetilde{K}_{33}\eta_3 + 2\,\theta_o^2\gamma_1)}{\gamma_1[3\,\eta_3 - \theta_o^2(3\,\eta_2 + 11\,\eta_3)]}, \tag{24}$$

the unique real solution of Eq. (21) is given by $k^2 \equiv 0$.

So, according to the leading mode principle, a homogeneous bending of the director (and, therefore, an absence of walls) would be found for these values of the magnetic field [19,20]. Some of us have done experimental investigations of these results and found that, no matter how close to the Freèdericksz threshold the measurements are made, magnetic walls were always found [6]. Furthermore, the time that the observer must wait for the clear observation of these structures seems to become infinite as the critical point is approached.

As these experimental results are in clear contradiction with the predictions of the leading mode principle, it becomes necessary to understand the formation of these walls from another point of view. In order to do it, we consider the second derivative, $\partial_k^2(\dot{b}_k)$, at $\tau=0$. It was analytically computed around the critical point, giving

$$\begin{aligned}\partial_k^2(\dot{b}_k) &\approx 2\,\eta_3^2[3\,\gamma_1(h^2-1)(\,\eta_3-\theta_o^2\,\eta_2)\\ &\quad -3\widetilde{K}_{33}\eta_3^2+\gamma_1\,\eta_3\,\theta_o^2(9-11h^2)]\\ &\approx 6\,\eta_3^4\widetilde{K}_{33}\frac{(h^2-h_{lm}^2)}{(1-h_{lm}^2)}.\end{aligned} \tag{25}$$

From this result, it is easy to see that, at the point $h=h_{lm}$, the second derivative is null and, furthermore, it changes sign when this point is crossed. Moreover, in the whole interval $1 \leqslant h^2 \leqslant h_{lm}^2$, the second derivative is small (it is proportional to η_3^4) and negative. Hence, as the range of the modes contributing to the walls formation is given by the inverse of $\partial_k^2(\dot{b}_k)$, the usual interpretation that follows Eq. (22), which says that in the forbidden region the unique mode contributing to the wall's formation is given by $\widetilde{k}^2=0$, cannot be true.

Likewise, a straightforward calculation shows that

$$\partial_k^3(\dot{b}_k)=0 \quad \text{at} \quad \tau=0 \quad \text{for} \quad 1 \leqslant h^2 \leqslant h_{lm}^2. \tag{26}$$

Finally, using an analytical computation, it was found that

$$\partial_k^4(\dot{b}_k) = -24\widetilde{K}\frac{\eta_1}{\eta_3} + O(\,\theta_o^2) < 0 \quad \text{at} \quad \tau=0 \quad \text{for} \quad h=h_{lm}, \tag{27}$$

where $O(\theta_o^2)$ represents the terms of the order of θ_o^2, or higher, that are insignificant. So, exactly at $h=h_{lm}$, only the fourth derivative of the growing speed of the director with relation to k is non-null and it is this value that controls the width of the distribution of modes at this point.

Above, we have argued that, in the forbidden region, the interaction between different modes does not need to be taken into account due to the fact that its coupling term is negligibly small, nor does it distinguish between different modes. Nevertheless, one can challenge this argument by saying that the long time evolved in the building of the observed structures may invalidate our reasoning. However, the result stated above, saying that in the forbidden region there is no leading mode and that all modes in the neighborhoods of $\widetilde{k}^2=0$ contribute to the final profile of the walls, was established in the very beginning of the process, at $\tau=0$, before the interaction between the different modes had time to produce any significant effect. So, the main message of this work, namely the breakdown of the leading mode principle in the forbidden region, does depend on these interactions.

The results exposed in Eqs. (22)–(27) are graphically exhibited in Fig. 1, where a numerical computation is shown using the known parameters of the (N-(p-methoxybenzylidene)-p'-butylaniline) (MBBA) for the behavior of $\dot{b}_{\widetilde{k}}$ as a function of $\widetilde{k}$ as the point h_{lm} is approached. From these figures, we clearly see that, around h_{lm}, the concept of an isolated mode contributing to the observed periodicity of the walls is meaningless and a new way to compute the contribution to the final periodicity of the walls must be found.

We will now propose a way to calculate the walls' periodicity in the forbidden region. As in this region the observed periodicity of the walls cannot be understood as a result of the contribution of a unique and isolated mode, we may suppose that the final walls' periodicity is determined by the long-time collective growth of the modes neighboring the mode $\widetilde{k}=0$. In this case, a natural candidate to fix the participation of each of these modes in the final profile is merely the maximum amplitude that each of them can attain. So, it can be assumed that, around $\widetilde{k}\simeq 0$, the final profile $\theta(x)=\theta(x,t\rightarrow\infty)$ of the walls could be approximated by

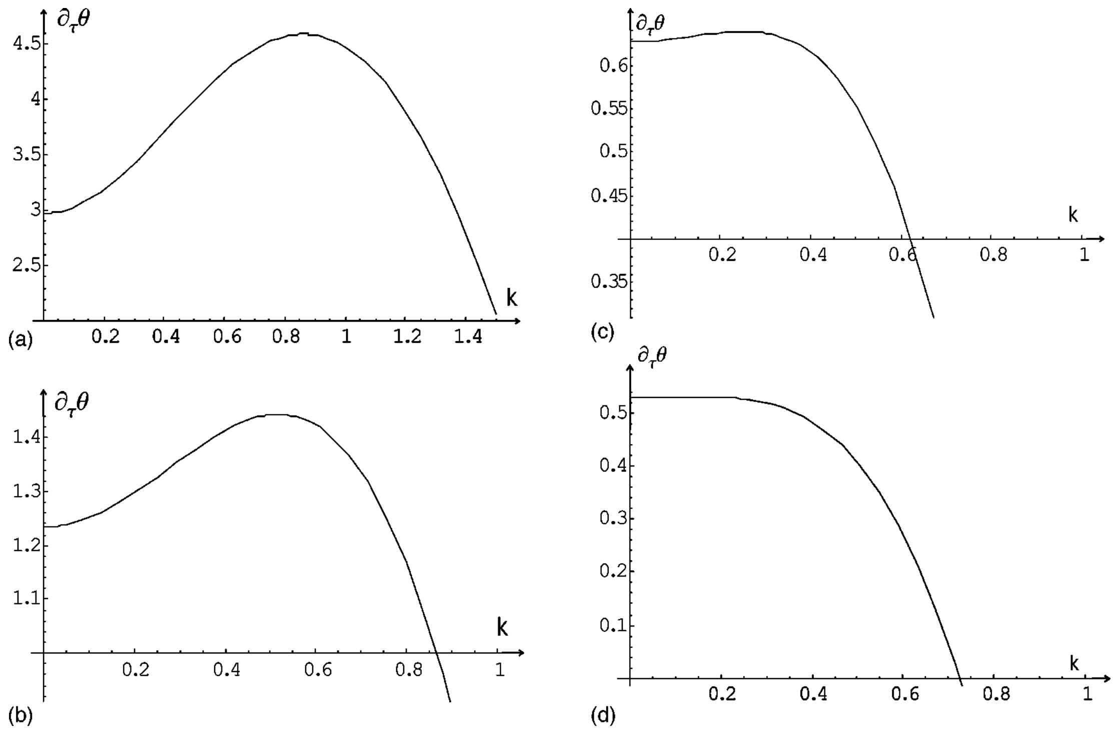

FIG. 1. Graphics of the function $\partial_\tau\theta$ at $\tau=0$ when $\tilde{k}$ is changed. Each figure was computed for a different h value. By using the known parameters of the MBBA, they show the evolution of the leading mode as the point h_{lm}, defined at Eq. (24), is approached. (a) was computed at $h=2h_{lm}$, and the point $\tilde{k}$, for which the function $\partial_\tau\theta$ has the maximum growth, is clearly observed. This is the fastest mode that, according to the leading mode principle, will determine the final periodicity of the wall. (b) was computed at $h=1.35h_{lm}$, and it shows that the leading mode is less pronounced. (c) was computed at $h=1.1h_{lm}$, showing that the leading mode is not clearly recognizable. (d) was computed at $h=h_{lm}$, demonstrating that the leading mode collapsed to the point $\tilde{k}=0$. From these figures we have clear evidence that, as h_{lm} is approached, the width of the modes around the leading mode becomes so large that an isolated leading mode cannot be recognized. The magnetic field was rescaled is such a way that the Frèedericksz threshold was taken as unity. The unities of $\tilde{k}$ and $\partial_\tau\theta$ can be considered as arbitrary.

$$\theta(x)=\lim_{t\to\infty}\sum_{\tilde{k}} b_{\tilde{k}}(t)\cos(\tilde{k}x)$$
$$\approx\int_0^{L/2}\theta_{\max}(\tilde{k})\cos(\tilde{k}x), \tag{28}$$

where $\theta_{\max}(\tilde{k})$ is the asymptotic limit of Eq. (19), given by Eq. (17). However, with the form for $\theta_{\max}(k)$ given by Eq. (17), this integration is extremely difficult, and probably impossible. As the maximum of $\theta_{\max}(k)$ occurs at $\tilde{k}=0$, it can be expanded around this point and

$$\theta_{\max}(k)\simeq\sqrt{a-bk^2} \tag{29}$$

is obtained, where

$$a=\frac{3}{2h^2}(h^2-1)\quad\text{and}\quad b=\frac{3}{2}\frac{\tilde{K}}{h^2}. \tag{30}$$

It has been assumed that $(h^2-1)^2$ is small, so

$$\theta(x)=\sqrt{b}\int_{-\sqrt{a/b}}^{\sqrt{a/b}}\sqrt{\frac{a}{b}-k^2}\cos(kx)dk$$
$$=\frac{\pi\sqrt{a}}{x}J_1\left(\sqrt{\frac{a}{b}}x\right), \tag{31}$$

where $J_1(\sqrt{a/b}x)$ is a first-class Bessel function of order 1. As, due to our nonlinear approximations, our results cannot be extended to the walls' nodes and consequently they are only valid for small x, we can use the approximation $J_1(x)\simeq(x/2)\cos(x/2)$ to obtain

$$\theta(x)\simeq\frac{\pi}{2}\frac{a}{\sqrt{b}}\cos\frac{1}{2}\sqrt{\frac{a}{b}}\,x$$
$$=\frac{\pi}{2}\sqrt{\frac{3}{2\tilde{K}h^2}}\,(h^2-1)\cos\frac{1}{2}\sqrt{\frac{(h^2-1)}{\tilde{K}}}\,x. \tag{32}$$

With this equation for the profile of the wall along the $\vec{e}_x$ direction, we can get its wave vector, which is given by

$$\tilde{k}^2 = \frac{1}{4\tilde{K}}(h^2 - 1), \tag{33}$$

which is just the result experimentally found, namely a linear relationship between h^2 and $\tilde{k}^2$ [6]. Furthermore, this expression for $\tilde{k}^2$ can be substituted in Eq. (16) to obtain the time spent in the walls' formation, in the forbidden region, as a function of the magnetic field h. The result is given by

$$\tau_o(h^2) = \frac{4}{3(h^2-1)}\left\{\frac{(h^2-1)(\eta_1-\gamma_1)+4\tilde{K}\eta_3}{(h^2-1)\eta_1+4\tilde{K}\eta_3}\right\}, \tag{34}$$

which, as was experimentally found, diverges when $h^2 \to 1$.

IV. CONCLUSION

In this paper, we have studied the formation of periodic patterns—magnetic walls—at the neighborhoods of the Frèedericksz critical point. These structures were induced by an external magnetic field and a twist-bend geometry was used [1]. We have shown that the results of the leading mode principle, which is the theory that up to now has been used to describe their formation [7,8], cannot explain its arising in this region. It is known that in the interval $1 \leqslant h^2 \leqslant h_{lm}^2$, where h_{lm}^2 is given in Eq. (24), the leading mode principle predicts that the fastest mode will correspond to $k=0$. This result corresponds to a homogeneous bending of the director and, therefore, to the absence of any texture. As this result was not observed in the experiment, we have here strong evidence that, at least around the Frèedericksz threshold, the leading mode principle deserves some reformulation. The main result presented by this work states that, in this region, the mode $k=0$ is the center of a large distribution of modes where each of them gives almost the same contribution to the final profile of the observed walls. In order to obtain a model for the magnetic walls formation, we have supposed that, due to the long time involved in this process, around the Frèedericksz threshold, each mode attains its maximum amplitude. These isolated contributions have been added and the final profile of the periodic walls has been obtained. Our results agree with the experimental findings and, at least for the twist-bend geometry, we have proved that there is not a forbidden region. We have also computed the time spent with the formation of these structures and we have experimentally found that, as the Frèedericksz threshold is approached, this time approaches the infinite.

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